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The thermal behaviour of oscillating gas bubbles

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Several aspects of the oscillations of a gas bubble in a slightly compressible liquid are discussed by means of a simplified model based on the assumption of a spatially uniform internal pressure. The first topic considered is the linear initial-value problem for which memory effects and the approach to steady state are analysed. Large-amplitude oscillations are studied next in the limit of large and small thermal diffusion lengths obtaining, in the first case, an explicit expression for the internal pressure, and, in the second one, an integral equation of the Volterra type. The validity of the assumption of uniform pressure is then studied analytically and numerically. Finally, the single-bubble model is combined with a simple averaged-equation model of a bubbly liquid and the propagation of linear and weakly nonlinear pressure waves in such a medium is considered.

1. Introduction

The considerable amount of theoretical information available on the radial dynamics of gas bubbles in liquids (for reviews see e.g. Plesset & Prosperetti 1977; Apfel 1981; Prosperetti 1984*a*, *b*) has, for the most part, been obtained with rather crude models for the calculation of the internal pressure. In particular, very frequent for the case of bubbles containing mostly an incondensible gas, is the so-called polytropic approximation, in which the internal pressure p is calculated from $p = p_0(R_0/R)^{3\kappa}$, where R is the bubble radius and κ is a polytropic index. The subscript zero indicates equilibrium values. A notable exception is the small-amplitude case, for which a more complete treatment based on the conservation equations is possible (Devin 1959; Chapman & Plesset 1971; Prosperetti 1977; Fanelli, Prosperetti & Reali 1981 *a*, *b*). These studies have demonstrated the complexity of the factors that determine the internal pressure. In the nonlinear case, the serious limitations of the polytropic approximation have been investigated theoretically by comparison with the more sophisticated theory summarized in §2 (Prosperetti, Crum & Commander 1988), and experimentally by a technique which is particularly sensitive to the energy dissipation during the radial motion (Crum & Prosperetti 1984). On the basis of experience with similar systems, one expects that the same sensitivity would be found in the study of the chaotic regime of oscillating bubbles, which has recently been the object of several investigations based on the polytropic approximation (Lauterborn & Suchla 1984; Lauterborn & Parlitz, 1988; Smereka, Birnir & Banerjee 1987). Indeed, recent work shows that the predictions of the polytropic model are very different from those of a more refined one (Kamath & Prosperetti 1989).

These considerations motivate an interest in the model studied in this paper, which is essentially that originally proposed in a series of papers by Nigmatulin and co-workers (Nigmatulin & Khabeev 1984, 1977; Nagiev & Khabeev 1979; Nigmatulin, Khabeev & Nagiev 1981). The basic simplification consists of the approximation of a spatially uniform internal pressure. With this approximation, for a perfect gas, the three conservation equations for mass, momentum, and energy can be reduced to only one. Thus, even though numerical methods are required in general, the computational effort is very considerably reduced. Some results obtained with this model have been presented in earlier papers (Prosperetti *et al.* 1988; Kamath & Prosperetti 1989).

The present study has several purposes. First, the linear initial-value problem will be studied with particular consideration of the transient behaviour of oscillating bubbles (§3). Secondly, the nonlinear bubble response in the limiting cases of nearly isothermal and nearly adiabatic behaviour, in which explicit expressions for the internal pressure can be obtained, will be examined (§§4 and 5). Thirdly, the validity of the basic approximation of the model, namely the assumption of a spatially uniform internal pressure, is examined (§6). Lastly, the single-bubble results are integrated in a simple model of a bubbly liquid and applied to the study of linear and weakly nonlinear pressure waves in such media. The rich mathematical structure of the model is reflected in several previously unnoticed features of the resulting wave equations (§7). For completeness, a summary of the model is included in the next section.

2. Mathematical model

In all of the following we assume the bubble to maintain a spherical shape. While this is only an approximation, it may be noted that the thermal processes of concern in the present study are mainly influenced by the volume variations of the bubble and by the heat exchange with the liquid. Hence one would not expect a large effect of relatively small deviations from sphericity.

The enthalpy equation for a perfect gas may be written

$$\rho C_p \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}p}{\mathrm{d}t} = \nabla \cdot (K \nabla T), \tag{2.1}$$

where ρ is the density, C_p the specific heat at constant pressure, K the thermal conductivity, and $\mathrm{d}/\mathrm{d}t$ the convective derivative. This equation may be written in an alternative form by combining it with the equation of continuity, with the result

$$\frac{\mathrm{d}p}{\mathrm{d}t} + \gamma p \nabla \cdot \boldsymbol{u} = (\gamma - 1) \nabla \cdot (K \nabla T), \tag{2.2}$$

where γ is the ratio of specific heats, $\boldsymbol{u}$ the velocity field, and the relation $C_p \rho T = \gamma p/(\gamma - 1)$ valid for perfect gases has been used. From a consideration of the momentum equation, it is easy to estimate that the maximum pressure difference Δp between any two points in the bubble satisfies

$$\frac{\Delta p}{p} \sim O\left(Ma \frac{R}{\lambda}, Ma^2\right), \tag{2.3}$$

where λ is a typical wavelength in the gas and $Ma = \dot{R}/c$ is the Mach number of the bubble wall motion referred to the speed of sound in the gas c. In many applications $R \ll \lambda$, $Ma \ll 1$, so that the pressure in the bubble may be taken to be spatially uniform to a good approximation. The validity of this assumption is examined in §6. Here we proceed to its exploitation in (2.2) which can then be integrated immediately to find the following expression for the radial velocity field of the gas:

$$u = \frac{1}{\gamma p}\left[(\gamma-1)K\frac{\partial T}{\partial r} - \tfrac{1}{3}r\dot{p}\right]. \tag{2.4}$$

Upon imposing the kinematic boundary condition $u = \dot{R}$ at $r = R$, the following differential equation for the pressure is obtained:

$$\dot{p} = \frac{3}{R}\left[(\gamma-1)K\frac{\partial T}{\partial r}\bigg|_R - \gamma p\dot{R}\right]. \tag{2.5}$$

The temperature field must of course still be obtained from (2.1), in which the velocity appearing in the convective derivative is given by (2.4). In the present paper we shall solve this equation subject to the boundary condition

$$T(r = R(t), t) = T_0, \tag{2.6}$$

where T_0 is the undisturbed liquid temperature. This is of course an approximation, which is however justified for gas bubbles when vapour effects are negligible as discussed in greater detail in Prosperetti (1986) and Prosperetti *et al.* (1988).

For numerical work, and part of the following applications of this model, it is expedient to deal with a fixed boundary, which can be obtained by introducing the coordinate

$$y = r/R(t), \tag{2.7}$$

in terms of which the energy equation (2.1) becomes

$$\frac{\gamma}{\gamma-1}\frac{p}{T}\left(\frac{\partial T}{\partial t} + \frac{u-\dot{R}y}{R}\frac{\partial T}{\partial y}\right) = \dot{p} + \frac{1}{R^2}\nabla_y\cdot(K\nabla_y T), \tag{2.8}$$

where ∇_y indicates differentiation with respect to y. This subscript will be omitted henceforth. By use of (2.4) and (2.5), it is readily found that

$$u - \dot{R}y = \frac{\gamma-1}{\gamma p R}\left(K\frac{\partial T}{\partial y} - y\left[K\frac{\partial T}{\partial y}\right]_{y=1}\right). \tag{2.9}$$

We shall make use of dimensionless variables indicated by an asterisk and defined as follows:

$$\left.\begin{aligned} t &= \omega t_*, & R &= R_0 R_*, & T &= T_0 T_*, \\ p &= p_0 p_*, & K &= K_0 K_*, & u &= \omega R_0 u_*, \end{aligned}\right\} \tag{2.10}$$

where ω is the inverse of a characteristic time and the index 0 denotes undisturbed values. With these definitions (2.5), (2.8) and (2.9) become

$$\frac{p_*}{T_*}\left(\frac{\partial T_*}{\partial t_*} + \frac{u_* - R'_* y}{R_*}\frac{\partial T_*}{\partial y}\right) = \frac{\gamma-1}{\gamma}p'_* + \frac{D}{R_*^2}\nabla\cdot(K_*\nabla T_*), \tag{2.11}$$

$$u_* - R'_* y = \frac{D}{R_* p_*}\left(K_*\frac{\partial T_*}{\partial y} - y\frac{\partial T_*}{\partial y}\bigg|_{y=1}\right), \tag{2.12}$$

$$p'_* = \frac{3\gamma}{R_*}\left(\frac{D}{R_*}\frac{\partial T_*}{\partial y}\bigg|_{y=1} - p_* R'_*\right), \tag{2.13}$$

where the prime denotes differentiation with respect to the dimensionless time and

$$D = \frac{\chi}{\omega R_0^2}, \tag{2.14}$$

with $\chi = K_0/\rho_0 C_p$ indicating the thermal diffusivity, is the square of the ratio of the thermal penetration length $(\chi/\omega)^{\frac{1}{2}}$ to the undisturbed radius.

To close the model an equation for the bubble radius is necessary. For an incompressible liquid this equation has the well-known form

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{1}{\rho_L}\{p_B - p_\infty[1+S(t)]\}, \tag{2.15}$$

where ρ_L is the liquid density, p_∞ the static pressure, $p_\infty S$ the variable pressure at large distance from the bubble, and p_B the liquid pressure at the interface related to the internal pressure p by

$$p = p_B + \frac{2\sigma}{R} + 4\mu_L\frac{\dot{R}}{R}, \tag{2.16}$$

with σ the surface tension and μ_L the liquid viscosity. An equation approximately accounting for the liquid compressibility has been obtained by Keller (Keller & Kolodner 1956; Keller & Miksis 1980; Prosperetti & Lezzi 1986) and is

$$\left(1-\frac{\dot{R}}{c_L}\right)R\ddot{R} + \frac{3}{2}\left(1-\frac{1}{3}\frac{\dot{R}}{c_L}\right)\dot{R}^2 = \frac{1}{\rho_L}\left(1-\frac{\dot{R}}{c_L}+\frac{R}{c_L}\frac{d}{dt}\right)\{p_B - p_\infty[1+S(t)]\}, \tag{2.17}$$

where c_L is the speed of sound in the liquid. Consideration of the compressible case clarifies the meaning of $p_\infty S(t)$ and shows that this quantity is to be understood as the perturbation pressure at the location of the bubble if the bubble were absent (Keller & Miksis 1980; Lezzi & Prosperetti 1987).

By combining (2.15) and (2.17) – or by applying $[1-(\dot{R}/c_L)+(R/c_L)\,d/dt]^{-1}$ to both sides of (2.17) and expanding in powers of c_L^{-1} – it is possible to derive the equation (Prosperetti 1987)

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 - \frac{1}{c_L}(R^2\dddot{R} + 6R\dot{R}\ddot{R} + 2\dot{R}^3) = \frac{1}{\rho_L}\{p_B - p_\infty[1+S(t)]\}, \tag{2.18}$$

which will also be useful in the following. At first sight this equation exhibits the odd feature of requiring three initial conditions for its integration. Actually, this is just an artifact of the perturbation scheme used for its derivation (Prosperetti & Lezzi 1986; Lezzi & Prosperetti 1987) and is of no conceptual significance. To the same order of accuracy in the effects of the compressibility of the liquid, an initial condition for $\ddot{R}$ can be obtained by substituting the given initial conditions for R and $\dot{R}$ into (2.15). For a constant ambient pressure (i.e. $S = 0$) all these equations have the equilibrium solution

$$p_0 = p_\infty + \frac{2\sigma}{R_0}.$$

The dimensionless form of the radial equation (2.17) is

$$\left(1-\frac{R'_*}{c_*}\right)R_*R''_* + \frac{3}{2}\left(1-\frac{R'_*}{3c_*}\right)R'^2_* = Z\left(1-\frac{R'_*}{c_*}+\frac{R_*}{c_*}\frac{d}{dt_*}\right)\{p_{B*} - (1-w)[1+S(t_*)]\}, \tag{2.19}$$

where
$$Z = \frac{p_0}{\omega^2 R_0^2 \rho_{\mathrm{L}}}, \quad c_* = \frac{c_{\mathrm{L}}}{\omega R_0}, \quad w = \frac{2\sigma}{R_0 p_0}, \tag{2.20}$$

while for the interface condition (2.16) we have

$$p_* = p_{\mathrm{B}*} + \frac{w}{R_*} + 2M\frac{R'_*}{R_*}, \tag{2.21}$$

with
$$M = \frac{2\mu_{\mathrm{L}}\omega}{p_0}. \tag{2.22}$$

In the numerical examples that will be shown in this paper we have used

$$S(t_*) = -\epsilon \sin t_*, \tag{2.23}$$

with ϵ a dimensionless acoustic amplitude.

3. Small-amplitude motion

In the case of small-amplitude motion, (2.11) and (2.13) become

$$\frac{\partial T_*}{\partial t_*} = \frac{\gamma - 1}{\gamma} p'_* + D\nabla^2 T_*, \tag{3.1}$$

$$p'_* = 3\gamma\left(D\frac{\partial T_*}{\partial y}\bigg|_1 - R'_*\right). \tag{3.2}$$

We set
$$R_* = 1 + X(t_*), \quad p_* = 1 + P(t_*), \quad T_* = 1 + \Theta(t_*), \tag{3.3}$$

and take Laplace transforms, indicated by a tilde. The transformed equations are readily solved and the pressure is found in the form

$$\tilde{P} = -\tilde{F}\left(\frac{s}{D}\right)\tilde{X} + \frac{1}{s}\tilde{Q}_{\mathrm{i}}. \tag{3.4}$$

In this expression s is the variable conjugated to the dimensionless time, the function $\tilde{F}$ is defined by

$$\tilde{F}(z) = \frac{3\gamma z}{z + 3(\gamma - 1)(z^{\frac{1}{2}} \coth z^{\frac{1}{2}} - 1)}, \tag{3.5}$$

and the function $\tilde{Q}_{\mathrm{i}}$, given by

$$\tilde{Q}_{\mathrm{i}} = P_{\mathrm{i}} + \tilde{F}\left(\frac{s}{D}\right)\left[X_{\mathrm{i}} - \int_0^1 \frac{\sinh (s/D)^{\frac{1}{2}} y}{\sinh (s/D)^{\frac{1}{2}}} y\Theta_{\mathrm{i}}(y)\, \mathrm{d}y\right], \tag{3.6}$$

accounts for the effect of the initial conditions (index i). In spite of the presence of square roots, it is not necessary to introduce branch cuts in the definition of the functions appearing in (3.4).

It is useful to consider these results in conjunction with the equation of motion for the bubble boundary. In the present context the third-order equation (2.18) proves most useful because it leads to somewhat simpler expressions. Upon non-dimensionalization and linearization it becomes

$$X'' - \frac{X'''}{c_*} + 2ZMX' - ZwX = Z(P - S). \tag{3.7}$$

The term MX' accounts for energy dissipation by viscosity, while the acoustic losses are described by the term X'''/c_*, as will be seen below.

It is readily established that, for $z \to 0$,

$$\tilde{F} \sim 3 + \frac{\gamma - 1}{5\gamma} z, \tag{3.8}$$

so that, for $s \to 0$ (i.e. $t \to \infty$) we find

$$\tilde{Q}_i \to P_i + 3\left(X_i - \int_0^1 y^2 \Theta_i \,\mathrm{d}y\right). \tag{3.9}$$

The condition of mass conservation for the entire bubble, in the present dimensionless units, is

$$3R^3 p \int_0^1 y^2 T^{-1} \,\mathrm{d}y = 1. \tag{3.10}$$

Linearization of this relation and evaluation at $t = 0$ shows that the right-hand side of (3.9) vanishes, so that the large-time behaviour of the bubble is not affected by the initial conditions, as expected. From (3.4) it is further seen that the (dimensionless) timescale for this to happen is of the order of D^{-1}, i.e. of the thermal penetration time over the bubble radius. We shall not consider initial effects further, and we shall put $\tilde{Q}_i = 0$ in the following.

The primitive of the transform (3.5) cannot be written down in a transparent form, but certain interesting conclusions can nevertheless be derived from (3.5) directly. Neglecting the contribution $\tilde{Q}_i$ of the initial conditions and using the convolution theorem, we may write

$$P(t) = -\int_0^t F(Dt') X(t-t') \,\mathrm{d}t'. \tag{3.11}$$

When this result is substituted into (3.7), it is seen that the equation takes on an integro-differential structure. This memory effect is due to the diffusive nature of the process that determines $P(t)$.

Let us now consider the case in which D is large, i.e. the bubble is small compared with the characteristic thermal penetration length. Then the argument Dt' is large over most of the integration range in (3.11), and we may make use of the asymptotic form of F for large argument. Using (3.8) we find

$$F(t) \sim 3\delta(t) - \frac{\gamma - 1}{5\gamma} \delta'(t),$$

from which, upon substitution into (3.11),

$$P \sim -3X - \frac{\gamma - 1}{5\gamma D} X'. \tag{3.12}$$

The memory effect is seen to disappear in this limit and, if the last term is disregarded, this result is just the linearized form of the isothermal relation $pR^3 = 1$. The equation of motion (3.7) takes the form

$$X'' - \frac{X'''}{c_*} + 2Z\left(M + \frac{\gamma - 1}{10\gamma D}\right) X' + (3 - w) X = -ZS. \tag{3.13}$$

In dimensional terms the quantity $3-w$ is given by

$$\omega_0^2 = \frac{1}{R_0^2}\left(3p_0 - \frac{2\sigma}{R_0}\right), \tag{3.14}$$

and thus coincides with Minnaert's (1933) result for the natural frequency of an isothermal bubble. The second term multiplying X' accounts for the thermal damping in this limit and, again in dimensional form, is given by

$$\beta_{\mathrm{th}} = \frac{\gamma-1}{10\gamma}\frac{p_0}{\rho_{\mathrm{L}}\chi}, \tag{3.15}$$

which is also a known result (Prosperetti 1984*a*).

In the opposite limit of very small D or, for any D, up to times such that $Dt \ll 1$, the function F is determined by the behaviour of $\tilde{F}(z)$ for large z, which is

$$\tilde{F}(z) \sim 3\gamma[1-3(\gamma-1)z^{-\frac{1}{2}}].$$

In this case (3.11) reduces to

$$P(t) = -3\gamma X(t) + 9(\gamma-1)\gamma\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\int_0^t X(t')\frac{\mathrm{d}t'}{(t-t')^{\frac{1}{2}}}, \tag{3.16}$$

which, with the neglect of the last term, is just the linearization of the adiabatic relation $pR^{3\gamma} = 1$. We conclude that the initial motion of every bubble is adiabatic, even though the subsequent behaviour may be very different. This remark has a bearing on the normal-mode method for the calculation of the bubble's natural frequency during free oscillations (see e.g. Chapman & Plesset 1971). In this approach one assumes at the outset an exponential time dependence, the complex frequency of which is obtained as an eigenvalue of the problem. As has been found in other problems in which the dynamics is governed by diffusive processes (see e.g. Prosperetti 1980, 1981), this approach gives the *asymptotic* time dependence, but does not fully describe the transient. In particular, the imposition of initial conditions on the solution obtained in this way may be incorrect.

Another interesting situation arises when the bubble is driven by an external sound field into oscillations which eventually become steady. To investigate this long-term behaviour, which with the present non-dimensionalization takes place at a dimensionless frequency of unity, we multiply (3.11) by $\exp(-\mathrm{i}t)$ and find

$$P\,\mathrm{e}^{-\mathrm{i}t} = -\int_0^t [F(Dt')\,\mathrm{e}^{-\mathrm{i}t'}][\mathrm{e}^{-\mathrm{i}(t-t')}X(t-t')]\,\mathrm{d}t', \tag{3.17}$$

the Laplace transform of which is

$$\tilde{P}(s+\mathrm{i}) = -\tilde{F}\left(\frac{s+\mathrm{i}}{D}\right)\tilde{X}(s+\mathrm{i}). \tag{3.18}$$

For an oscillatory motion proportional to $\exp(\mathrm{i}t)$, as $t\to\infty$,

$$\mathrm{e}^{-\mathrm{i}t}X(t)\to X_{\mathrm{ss}}, \quad \mathrm{e}^{-\mathrm{i}t}P(t)\to P_{\mathrm{ss}},$$

where X_{ss} and P_{ss} are complex constants representing the steady-state amplitudes of the radial oscillations and of the internal pressure perturbations. Hence, taking the limit $s\to 0$ in (3.18), we find

$$P_{\mathrm{ss}} = -\tilde{F}\left(\frac{\mathrm{i}}{D}\right)X_{\mathrm{ss}}, \tag{3.19}$$

which, in terms of $P(t)$ and $X(t)$, may formally be written

$$P = -\operatorname{Re}\tilde{F}(\mathrm{i}/D)\,X - \operatorname{Im}\tilde{F}(\mathrm{i}/D)\,X'. \tag{3.20}$$

Upon also setting $X''' = -\mathrm{i}X'$, the radial equation (3.7) can be cast in the standard form of a driven harmonic oscillator, namely

$$X'' + 2\left[ZM + \frac{1}{c_*} + \tfrac{1}{2}Z\operatorname{Im}\tilde{F}(\mathrm{i}/D)\right]X' + Z\,[\operatorname{Re}\tilde{F}(\mathrm{i}/D) - w]\,X = -ZS. \tag{3.21}$$

However, it is important to stress that this analogy holds only asymptotically for times $t \gg D^{-1}$, so that only in this limit does a driven oscillating bubble behave as a standard oscillator. The temperature field in the bubble in this asymptotic stage is given by

$$\Theta = \frac{\gamma-1}{\gamma}P\left[1 - \frac{\sinh(\mathrm{i}/D)^{\frac{1}{2}}y}{y\sinh(\mathrm{i}/D)^{\frac{1}{2}}}\right]. \tag{3.22}$$

On the basis of (3.21), one can define the natural frequency of the bubble to be $Z(\operatorname{Re}\tilde{F} - w)$ or, in dimensional terms,

$$\omega_0^2 = \frac{p_0}{\rho_L R_0^2}(3\kappa - w), \tag{3.23}$$

where the effective polytrophic index κ is defined by

$$\kappa = \tfrac{1}{3}\operatorname{Re}\tilde{F}(\mathrm{i}/D). \tag{3.24}$$

Since the quantity D depends on the frequency, this equation is actually implicit in ω_0. The oscillations are damped by viscosity, acoustic radiation, and thermal effects and the corresponding damping constants, again in dimensional terms, are

$$\beta_v = \frac{2\mu}{\rho_L R_0^2}, \tag{3.25}$$

$$\beta_{ac} = \frac{\omega^2 R_0}{2c_L}, \tag{3.26}$$

$$\beta_{th} = \frac{p_0}{2\rho_L \omega R_0^2}\operatorname{Im}\tilde{F}\left(\frac{\mathrm{i}}{D}\right). \tag{3.27}$$

A more explicit expression for $\tilde{F}(\mathrm{i}/D)$ is

$$\tilde{F}\left(\frac{\mathrm{i}}{D}\right) = \frac{3\gamma\eta^2}{\eta[\eta + 3(\gamma-1)A_-] - 3\mathrm{i}(\gamma-1)(\eta A_+ - 2)}, \tag{3.28}$$

where
$$\eta = \left(\frac{2}{D}\right)^{\frac{1}{2}} = R_0\left(\frac{2\omega}{\chi}\right)^{\frac{1}{2}}, \quad A_\pm = \frac{\sinh\eta \pm \sin\eta}{\cosh\eta - \cos\eta}. \tag{3.29}$$

The variable η may be considered either as a dimensionless radius for a fixed forcing frequency, or as a dimensionless frequency for a fixed radius. These results show therefore that, unlike the case of an ordinary oscillator, neither the effective damping nor the resonant frequency are constant but depend on the driving frequency.

Figure 1 shows a graph of κ defined by (3.24) as a function of η for monatomic

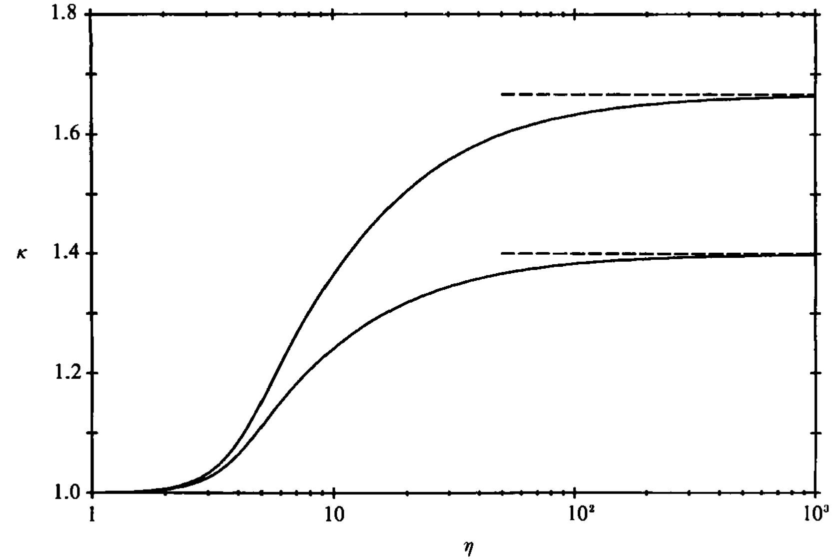

FIGURE 1. The effective polytropic index κ for linear oscillations as a function of the dimensionless driving frequency η defined in (3.29) for monatomic ($\gamma = \frac{5}{3}$) and diatomic ($\gamma = \frac{7}{5}$) gases. The dashed horizontal lines indicate the adiabatic limit $\kappa = \gamma$.

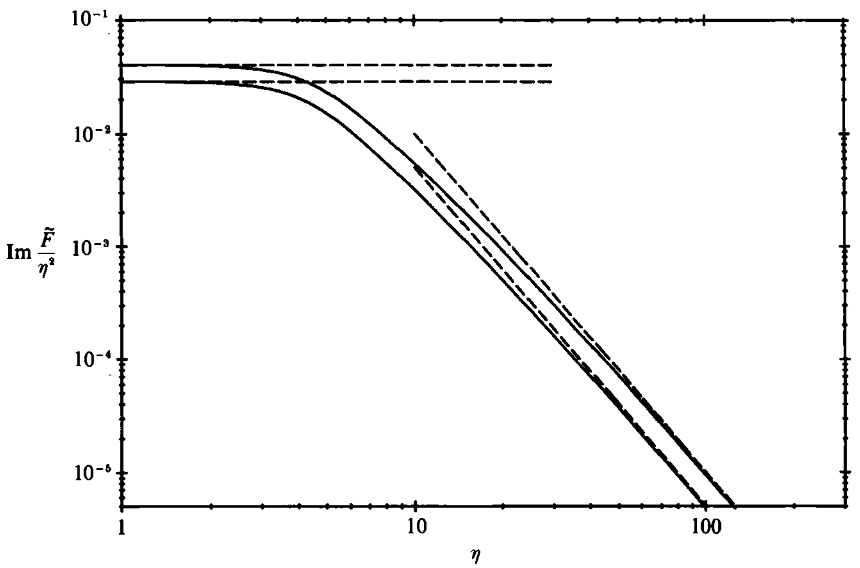

FIGURE 2. The dimensionless thermal damping function $\mathrm{Im}\,\tilde{F}/\eta^2$ defined by (3.28) as a function of the dimensionless driving frequency η defined in (3.29) for monatomic ($\gamma = \frac{5}{3}$, upper line) and diatomic ($\gamma = \frac{7}{5}$, lower line) gases. The dashed lines are the first term in the approximations (3.30), (3.31)

($\gamma = \frac{5}{3}$) and diatomic ($\gamma = \frac{7}{5}$) gases. The expected behaviour at low frequency, $\kappa \to 1$, and at high frequency, $\kappa \to \gamma$, are clear from this figure. Figure 2 is a plot of $\mathrm{Im}\,\tilde{F}/\eta^2$ as a function of η. The asymptotic behaviour is readily obtained from (3.28) and is, for $\eta \to 0$,

$$\kappa \to 1 + \frac{\gamma - 1}{90\gamma}\left(\frac{1}{7} - \frac{\gamma - 1}{10\gamma}\right)\eta^4, \quad \mathrm{Im}\,\tilde{F} \to \frac{\gamma - 1}{10\gamma}\eta^2, \tag{3.30}$$

while, for $\eta \to \infty$,

$$\kappa \to \gamma\left[1-\frac{3(\gamma-1)}{\eta}+O(\eta^{-3})\right], \quad \operatorname{Im}\tilde{F} \to 9\gamma\frac{\gamma-1}{\eta}\left[1-\frac{2}{\eta}+O(\eta^{-3})\right]. \tag{3.31}$$

The dashed lines in figure 2 show the leading-order term in the asymptotic relations for $\operatorname{Im}\tilde{F}$.

4. The nearly isothermal case

When the thermal penetration length is large compared with the radius, one expects the temperature in the bubble to deviate little from the equilibrium value. In this case the parameter D defined in (2.14) is large and a perturbation solution in D^{-1} may be attempted.

Combining (2.11)–(2.13) we may write

$$\frac{p}{T}\left[\frac{\partial T}{\partial t}+\frac{D}{pR}\left(K\frac{\partial T}{\partial y}-y\left.\frac{\partial T}{\partial y}\right|_1\right)\frac{\partial T}{\partial y}\right] = 3\frac{\gamma-1}{R^2}\left(D\left.\frac{\partial T}{\partial y}\right|_1-pRR'\right)+\frac{D}{R^2}\nabla\cdot(K\nabla T). \tag{4.1}$$

Here we still use dimensionless variables, but asterisks have been dropped for convenience.

We now expand

$$T = 1+D^{-1}T_1+D^{-2}T_2,\ldots, \quad K(T) = 1+D^{-1}K_0'T_1+\ldots, \tag{4.2}$$

$$p = R^{-3}(1+D^{-1}G_1+D^{-2}G_2+\ldots), \tag{4.3}$$

where $K_0' = [\mathrm{d}K/\mathrm{d}T]_0$ and the isothermal relation $pR^3 = 1$ has been anticipated in the zero-order term of the expansion for p. For the time being R is to be considered as prescribed. In terms of the same expansions, the statement (3.10) of mass conservation is

$$1 = 1+D^{-1}\left(G_1-3\int_0^1 y^2T_1\,\mathrm{d}y\right)+D^{-2}\left[G_2-3G_1\int_0^1 y^2T_1\,\mathrm{d}y-3\int_0^1 y^2(T_2-T_1^2)\,\mathrm{d}y\right]+\ldots. \tag{4.4}$$

We shall also need the differential equation (2.13) for the pressure, repeated here for convenience:

$$p' = \frac{3\gamma}{R}\left(\frac{D}{R}\left.\frac{\partial T}{\partial y}\right|_1-pR'\right). \tag{4.5}$$

Upon substitution of (4.2) and (4.3) into (4.1), the equation for T_1 is found to be

$$\nabla^2T_1 = 3(\gamma-1)\left(\frac{R'}{R^2}-\left.\frac{\partial T_1}{\partial y}\right|_1\right).$$

The solution satisfying $T_1 = 0$ at $y = 1$ is

$$T_1 = \tfrac{1}{2}(\gamma-1)\left(\frac{R'}{R^2}-\left.\frac{\partial T_1}{\partial y}\right|_1\right)(y^2-1),$$

from which one obtains that, for consistency,

$$\left.\frac{\partial T_1}{\partial y}\right|_1 = \frac{\gamma-1}{\gamma}\frac{R'}{R^2}, \tag{4.6}$$

so that

$$T_1 = \frac{\gamma-1}{2\gamma}\frac{R'}{R^2}(y^2-1). \tag{4.7}$$

Upon substitution of (4.6) into the differential equation for the pressure (4.5) one confirms the correctness of the zero-order relation $pR^3 = 1$. Because the factor D multiplies the temperature gradient in (4.5), the first-order correction to this relation requires knowledge of T_2. However, the global mass conservation relation (4.4) can be used together with (4.7) to find

$$G_1 = -\frac{\gamma-1}{5\gamma}\frac{R'}{R^2}, \tag{4.8}$$

so that, to this order,

$$p = R^{-3}\left(1-\frac{\gamma-1}{5\gamma D}\frac{R'}{R^2}\right). \tag{4.9}$$

For small-amplitude motion this relation becomes

$$p = 1-3X-\frac{\gamma-1}{5\gamma D}X',$$

in agreement with the earlier result (3.12).

With due consideration of (4.7) and (4.8), the equation for T_2 simplifies to

$$\frac{1}{R}\frac{\partial T_1}{\partial t}+3(\gamma-1)\left[G_1\left.\frac{\partial T_1}{\partial y}\right|_1-\left.\frac{\partial T_2}{\partial y}\right|_1\right]+G_1\nabla^2 T_1-K_0'\nabla\cdot(T_1\nabla T_1) = \nabla^2 T_2.$$

Integrating and imposing consistency of the wall gradients as before we find

$$\left.\frac{\partial T_2}{\partial y}\right|_1 = \frac{\gamma-1}{\gamma}\left[\frac{R'}{R^2}G_1-\frac{1}{15\gamma R}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R'}{R^2}\right)\right]. \tag{4.10}$$

Upon substitution of this relation and of (4.6) into the pressure equation (4.5) one finds

$$G_1' = -\frac{\gamma-1}{5\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R'}{R^2},$$

which confirms the result (4.8). The final expression for T_2 is

$$T_2 = \frac{\gamma-1}{8\gamma R}(y^2-1)\left[\frac{1}{5}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R'}{R^2}\left(y^2-\frac{3\gamma+4}{3\gamma}\right)-\frac{\gamma-1}{\gamma}K_0'\frac{R'^2}{R^3}(y^2-1)-\frac{4}{5}\frac{\gamma-1}{\gamma}\frac{R'^2}{R^3}\right]. \tag{4.11}$$

With this result we can calculate the second-order pressure correction from (4.4), to find

$$G_2 = -\frac{1}{525}\left(\frac{\gamma-1}{\gamma}\right)^2\left[(2+15K_0')\frac{R'^2}{R^4}+\frac{12\gamma-7}{3(\gamma-1)}\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}t}\frac{R'}{R^2}\right]. \tag{4.12}$$

This expression is rather cumbersome and possibly of limited usefulness. We give it mainly to exhibit explicitly the large numerical factor in the denominator, which suggests that the first-order result (4.9) may be fairly accurate.

We can check this expectation directly by comparing some numerical results obtained from the complete formulation of §2 with the corresponding ones obtained from (4.9). As a preliminary, we plot in figure 3 a graph of the relation (2.14) defining D as a function of R_0 for a gas bubble in water at 1 bar. The middle line corresponds to $\omega = \omega_0$, the resonance frequency (3.23) of the linear theory. The upper line is for $\omega = 0.1\,\omega_0$ and the lower line for $\omega = 10\omega_0$. It is seen here that the interval of radii where the parameter D is large is rather restricted for frequency ranges where single-bubble phenomena are important, such as in acoustic cavitation. However, phenomena involving many bubbles, such as are encountered in bubbly liquids, often have characteristic frequencies much lower than those of the constituent bubbles. In this case it may be expected that (4.9) would be widely useful.

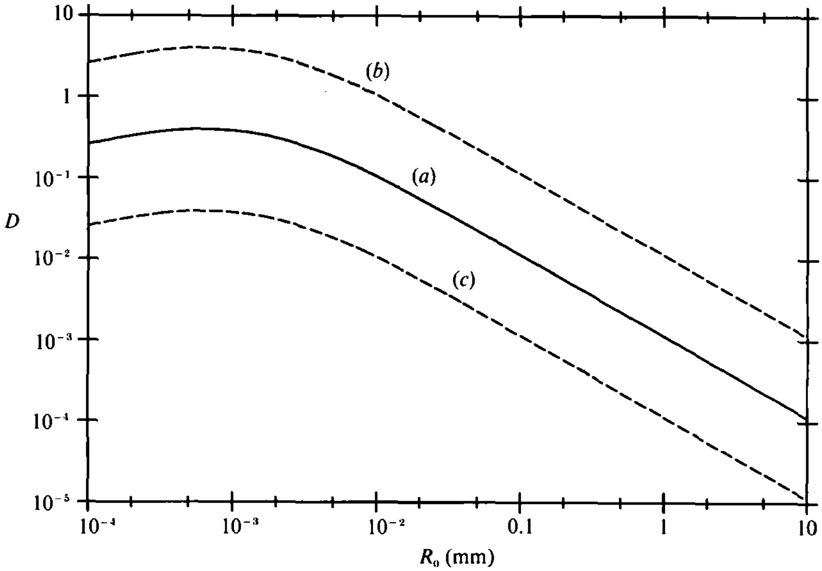

FIGURE 3. The dimensionless parameter D defined by (2.14) as a function of the bubble radius in mm for an air bubble in water at normal pressure. Line (a) corresponds to a frequency equal to the linear resonance frequency ω_0 given by (3.23), (b) $\omega = 0.1\omega_0$, and (c) $\omega = 10\omega_0$. The physical meaning of D is of the square of the ratio of the thermal penetration depth to the bubble radius.

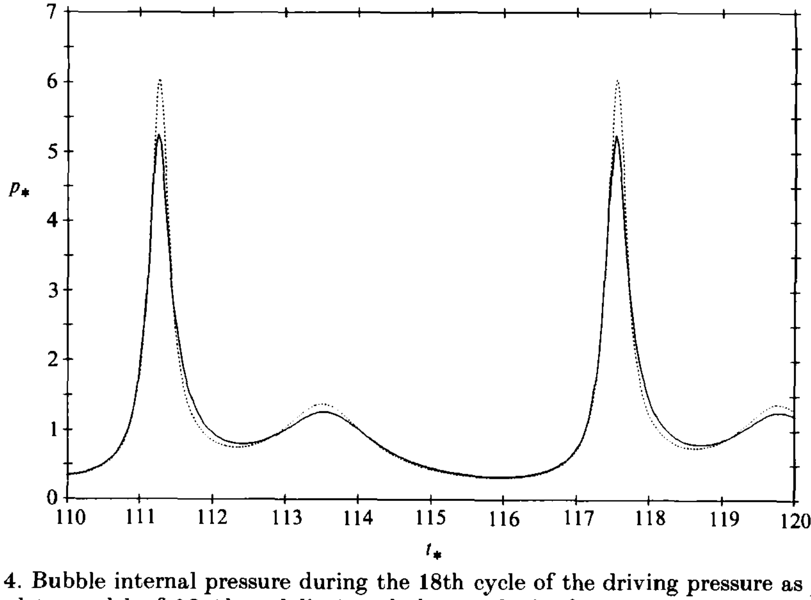

FIGURE 4. Bubble internal pressure during the 18th cycle of the driving pressure as predicted by the complete model of §2 (dotted line) and the nearly isothermal approximation (solid line). The equilibrium radius is 1 μm, the frequency $\omega/\omega_0 = 0.5$, the dimensionless pressure amplitude $\epsilon = 1$, $D = 1.096$, and the linear natural frequency is $\omega_0/2\pi = 3.89$ MHz.

Figure 4 refers to a 1 μm bubble (having a linear resonance frequency $\omega_0/2\pi = 3.89\,MHz$) driven by a sound field of frequency $\omega/\omega_0 = 0.5$ and dimensionless amplitude $\epsilon = 1$. The dotted line is the result of the complete model of §2 integrated by means of the spectral technique described in Kamath & Prosperetti (1989), while the solid line is the nearly isothermal approximation (4.9). In both cases, the initial conditions are $R(0) = 1$, $R'(0) = 0$ and the driving pressure field has the form (2.23). The quantity shown in the graph is the dimensionless internal pressure $p(t)$ during the 18th cycle of the sound field, by which time the transient has essentially died out and a steady regime of oscillation has been attained. Since the radius is obtained by integrating the internal pressure twice, the latter quantity is a much more sensitive indication of the accuracy of the approximation than the former one. A comparison of the radii, rather than the pressures, would hardly show any difference for this case. It can be seen that the error introduced by the expansion in powers of D^{-1} does not accumulate in time, but is limited to an underestimation of the peak internal pressure. This discrepancy is in the direction expected. Indeed, a characteristic feature of large-amplitude oscillations is the suddenness of the collapse phase. This implies that the timescale for the compressive heating of the gas is shorter than the sound period, so that the 'effective' value of D is smaller than anticipated on the basis of the linear theory. Hence, a greater heat transfer rate is predicted in the approximation than can actually occur, the bubble is colder than it should be, and the peak pressure smaller. Nevertheless, considering the fact that the value of D for this case is only $D = 1.096$, it may be concluded that the approximation works remarkably well.

In conclusion, it may be useful to give the dimensional form of the approximate pressure relation (4.9), which is

$$\frac{p}{p_0} = \left(\frac{R_0}{R}\right)^3 \left(1 - \frac{\gamma - 1}{5\gamma}\frac{R_0^3}{\chi}\frac{1}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t}\right). \tag{4.13}$$

5. The nearly adiabatic case

Another situation which is amenable to analytical investigation is that in which the parameter D is very small so that most of the gas in the bubble is thermally insulated from the liquid. This case has been investigated by Miksis & Ting (1984, 1987). We shall present here a different derivation of their results based on the model of §2.

To lowest order in $\delta = D^{\frac{1}{2}}$, the energy equation (2.11) reduces to

$$\frac{p}{T}\frac{\partial T}{\partial t} = \frac{\gamma - 1}{\gamma} p' + O(\delta^2). \tag{5.1}$$

For an initial condition of uniform temperature and equilibrium pressure, the solution is

$$T = p^{\gamma - 1/\gamma} + O(\delta^2), \tag{5.2}$$

i.e. the gas behaves adiabatically as expected. To the same approximation, the pressure equation (2.13) gives

$$pR^{3\gamma} = 1 + O(\delta^2). \tag{5.3}$$

Since (5.2) cannot satisfy the boundary condition $T = 1$ at $r = R$, a boundary layer must be present near the bubble surface. To resolve this layer we introduce the stretched variable

$$z = \frac{1 - y}{\delta},$$

and note that, from (2.12),

$$u - (1 - \delta z) R' = \frac{\delta}{pR}\left[(1 - \delta z)\frac{\partial T}{\partial z}\bigg|_{z=0} - K\frac{\partial T}{\partial z}\right] = O(\delta),$$

so that
$$u = R' + O(\delta)$$
in the boundary layer. Following now Plesset & Zwick (1952), we introduce the stretched Lagrangian variable
$$\eta = \frac{4\pi}{\delta}\int_r^R r^2 \rho \,\mathrm{d}r = 4\pi R^3 \int_0^z \rho \,\mathrm{d}z + O(\delta),$$
in terms of which the energy equation (2.11) becomes
$$\frac{r}{T}\left(\frac{\partial T}{\partial t}\right)_\eta = \frac{\gamma-1}{\gamma}p' + (4\pi)^2 R^4 \rho \frac{\partial}{\partial \eta}\left(K\rho\frac{\partial T}{\partial \eta}\right) + O(\delta).$$
The substitution
$$T = p^{\gamma-1/\gamma}\theta, \tag{5.4}$$
and the relation $\rho = p/T$ bring this into the form
$$\frac{\partial \theta}{\partial t} = (4\pi)^2 p R^4 \frac{\partial}{\partial \eta}\left(\frac{K}{T}\frac{\partial \theta}{\partial \eta}\right).$$
The new time variable
$$\tau = (4\pi)^2 \int_0^t p(t')\,R^4(t')\,\mathrm{d}t' \tag{5.5}$$
eliminates the function of time multiplying the right-hand side. Miksis & Ting (1984) further assume $K/T = 1$, which is justified on the basis of the corresponding approximation $\mu \propto T$ of compressible boundary-layer theory, together with the near constancy of the Prandtl number for many gases. Fokas & Yortsos (1982) have investigated the most general parabolic equation that can be reduced to the standard heat equation by a Bäcklund transformation, and on the basis of their results it appears that the case $K/T = 1$ is the only one of physical interest. Making then this assumption, we reduce the previous equation to
$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2}, \tag{5.6}$$
with the boundary conditions
$$\theta(\eta = 0, \tau) = p^{-(\gamma-1)/\gamma}, \quad \theta(\eta \to \infty, \tau) = 1,$$
which are a consequence of (2.6) and (5.4). The initial condition we shall consider is $\theta = 1$, which corresponds to the equilibrium state. The solution of this diffusion problem is immediate and in particular one finds
$$\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau [1 - p^{-(\gamma-1)/\gamma}(\tau - s)]\,(\pi s)^{-\frac{1}{2}}\,\mathrm{d}s. \tag{5.7}$$
In terms of (τ, η) the pressure equation (2.13) may be written
$$\frac{\mathrm{d}}{\mathrm{d}\tau}(pR^{3\gamma}) = -\frac{3\gamma\delta}{4\pi}R^{3(\gamma-1)}p^{(\gamma-1)/\gamma}\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} + O(\delta^2).$$
This relation shows that $pR^{3\gamma} = 1 + O(\delta)$, which can be used to express p in the right-hand side with the result
$$\frac{\mathrm{d}}{\mathrm{d}\tau}(pR^{3\gamma}) = -\frac{3\delta\gamma}{4\pi}\left.\frac{\partial \theta}{\partial \eta}\right|_{\eta=0} + O(\delta^2).$$

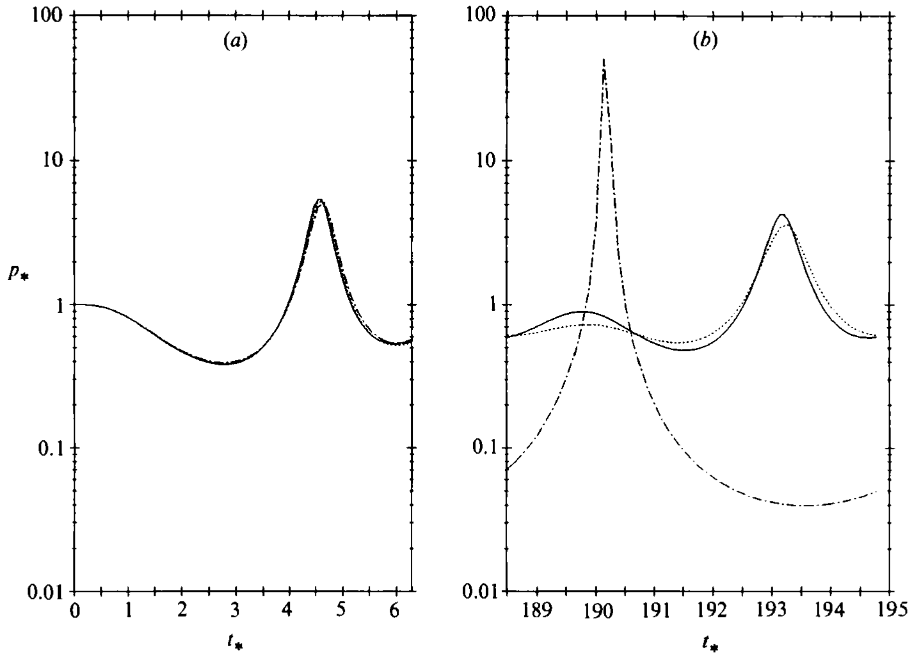

FIGURE 5. Internal pressure versus time during the first, (*a*) and 30th, (*b*) cycle of the driving pressure as predicted by the complete model of §2 (continuous line), the implicit nearly adiabatic approximation (5.5), (5.8) (dotted line), and the explicit nearly adiabatic approximation (5.10), (5.11) (dash-and-dot line). The equilibrium radius is 100 μm, the frequency $\omega/\omega_0 = 0.55$, and the dimensionless pressure amplitude $\epsilon = 0.5$. This is the same case as figure 6(*a*) of Miksis & Ting (1987). $D = 0.0208$ and the linear natural frequency $\omega_0/2\pi = 31.2$ kHz.

Upon substitution of (5.7) and integration subject to $R(0) = 1$, $p(0) = 1$, we find

$$pR^{3\gamma} = 1+\frac{3\gamma}{4\pi}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}[p^{-(\gamma-1)/\gamma}(\tau-s)-1]\,s^{-\frac{1}{2}}\,\mathrm{d}s+O(D). \tag{5.8}$$

The expression obtained by Miksis & Ting (1984) is

$$p^{1/\gamma}R^3 = 1+\frac{3}{4\pi}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}[p^{-(\gamma-1)/\gamma}(\tau-s)-1]\,s^{-\frac{1}{2}}\,\mathrm{d}s+O(D), \tag{5.9}$$

and, raised to the power γ, coincides with (5.8) up to order $D^{\frac{1}{2}}$. The memory effect expected on the basis of the linear analysis is apparent from these results.

To a consistent order in δ, one can obtain an explicit expression for the pressure by using, in the integral term, the lower-order approximation $p = R^{-3\gamma}$ to find

$$pR^{3\gamma} = 1+\frac{3\gamma}{4\pi}\left(\frac{D}{\pi}\right)^{\frac{1}{2}}\int_0^{\tau}[R^{3(\gamma-1)}(\tau-s)-1]\,s^{-\frac{1}{2}}\,\mathrm{d}s. \tag{5.10}$$

The same substitution can be made in the definition (5.5) of the auxiliary time variable τ to find, to the same approximation,

$$\tau = (4\pi)^2\int_0^t R^{4-3\gamma}(t')\,\mathrm{d}t'. \tag{5.11}$$

As before, we now examine the validity of these approximate results by a

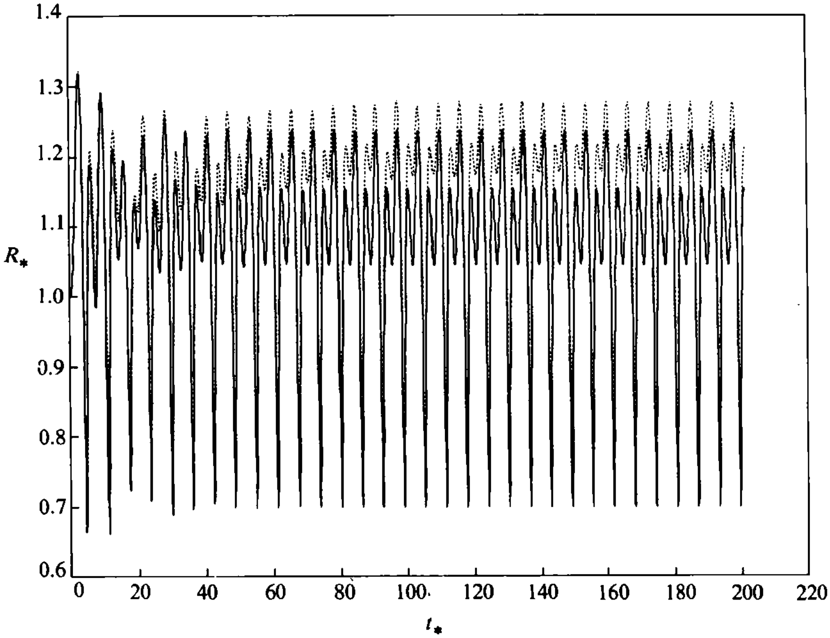

FIGURE 6. Radius versus time for the case of figure 5. The continuous line is the complete model of §2, the dotted line the implicit nearly adiabatic approximation.

comparison with the complete formulation of §2. From figure 3 it is seen that the parameter D tends to be rather small over a wide range of radii and frequencies. However, the expansion is in terms of $D^{\frac{1}{2}}$, rather than D, and this circumstance reduces the accuracy of the approximation. Figure 5 shows the internal pressure versus time in the forced oscillations of a 100 μm bubble during the first (figure 5a) and the thirtieth (figure 5b) cycle of the driving pressure for the same case shown in figure 6(a) of Miksis & Ting (1987). The linear resonance frequency of the bubble is $\omega_0/2\pi = 31.2$ kHz, and the sound field has a frequency $\omega/\omega_0 = 0.55$, so that $D = 0.0208$, $D^{\frac{1}{2}} = 0.144$. The dimensionless driving pressure amplitude is $\epsilon = 0.5$ and the initial conditions are $R(0) = 1$, $R'(0) = 0$. The continuous line is the result given by the complete model of §2, the dotted line is the implicit nearly adiabatic approximation (5.8), and the dash-and-dot line is the explicit nearly adiabatic approximation (5.10). It is seen that both approximations work fairly well during the first cycle. The explicit one, however, deteriorates rapidly owing to an accumulation of error and becomes useless after a few cycles. As for the implicit approximation, its difference with the complete model in figure 5(b) is somewhat obscured by the logarithmic ordinate scale, but it is fairly substantial. For example, the peak pressure is about 25% smaller than that given by the complete model. To get a better idea of the limits of this approximation, it is useful to consider the $R(t)$-curves for the two models, which we do in figure 6. It is seen here that the approximation (dotted line) predicts a larger bubble radius than the complete model (continuous line), consistently with the prediction of a lower pressure.

A striking result reported by Miksis & Ting (1987) is a marked increase of the average radius of the bubble. They found that in some cases the average radius stabilized after several tens of cycles while, in others, the gradual increase caused some sort of threshold to be exceeded and the bubble entered into very large-amplitude oscillations thereafter. To examine this process we have calculated the

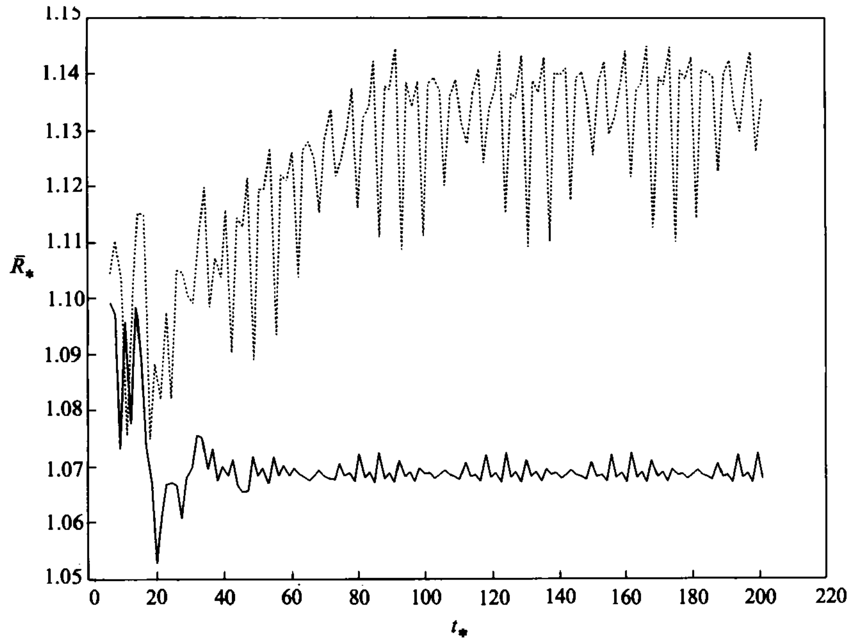

FIGURE 7. Running average of the radius from the previous figure. The continuous line is the complete model of §2, the dotted line the implicit nearly adiabatic approximation.

running average of $R(t)$ for the case of figure 6, and in figure 7 we compare the results of the complete model (continuous line) with those of the implicit nearly adiabatic approximation (dotted line). It is seen that the latter predicts a much greater increase than the former. This difference is probably due to an error in the total gas mass contained in the bubble arising as an artifact of the approximation. In other words, if one were to calculate the total mass contained in the bubble by carrying out the integration in (3.10) with the temperature field given by (5.2) and (5.6), one would find a result greater than 1. This hypothesis is based on the fact that the present model is very sensitive to mass errors. For example, our early numerical attempts with the complete model of §2 were plagued by apparent changes in the average radius that were invariably found to be associated with a loss or gain of mass arising, in that case, from numerical inaccuracies (Prosperetti *et al.* 1988). The spectral technique used for the calculations reported here conserves mass to approximately 1% in all the cases that we have tested.

Even though our calculations based on (5.8) produce an increase of the average radius greater than the complete model, this increase is substantially smaller than the one reported by Miksis & Ting (1987) for the same cases. The fact that their pressure equation has the form (5.9) rather than (5.8) is inconsequential as we have tested both forms and found indistinguishable results. We believe that their results are affected by a gradual accumulation of numerical error due to the use of an insufficiently accurate numerical technique. They used a three-point finite-difference interpolation of the pressure to calculate the integral (5.8) together with a predictor-corrector method. We have instead used cubic splines with iteration until convergence. Our experience is that the numerical treatment of this problem is quite delicate and that, for example, if linear interpolation is used instead of cubic splines, the results are quite different. Similarly, by relaxing somewhat the convergence test for the iterations of the time-stepping algorithm, we also found rapidly growing solutions. Use of the explicit approximation (5.10) had the effect of causing a greater average radius and occasional threshold-like phenomena similar to those reported by Miksis & Ting.

The previous comments are based on a number of tests that we have run of which figures 5–7 are one example. The results of a more extensive comparison will be presented in a separate study (Kamath, Oguz & Prosperetti 1990). We note however that this point is somewhat academic in that the mathematical structure of the nearly adiabatic approximation leads to a much more complicated computational problem than the complete formulation of §2, for which the spectral method is quite accurate and of straight-forward application. Furthermore, the integral over the past history of the pressure that appears in (5.8) requires longer computational times than the spectral method as soon as results for more than a few cycles are desired. On the other hand, computation times for the first few cycles are of the order of 2–3 minutes anyway and therefore, in practice, there is no compelling reason for selecting the adiabatic approximation over the exact formulation. While interesting, therefore, the approximate results (5.8) or (5.9) appear in the end to be of limited practical usefulness.

6. Accuracy of the uniform-pressure approximation

The approximation of uniform internal pressure used in §2 can be viewed in the context of a perturbation approach which is useful to judge its accuracy. Define

$$Ma^2 = \frac{(\omega R_0)^2 \rho_0}{p_0}, \tag{6.1}$$

where the subscript 0 indicates dimensional equilibrium quantities. Since the speed of sound in the gas is of order of $(p_0/\rho_0)^{\frac{1}{2}}$, the quantity defined by (6.1) is of the order of the Mach number in the gas for radial oscillations with an amplitude of order R_0. For a bubble at resonance, it is seen from (3.23) that

$$Ma^2 = (3\kappa - w)\frac{\rho_0}{\rho_L}, \tag{6.2}$$

which is a quantity of order 10^{-3} at normal temperature and pressure. This argument suggests the existence of a useful parameter range in which the oscillations of the bubble have a large amplitude, although the gas Mach number is small. It is easy to show that, in these conditions, the approximation of uniform pressure is quite justified.

With the definition (6.1) the dimensionless momentum equation in the gas becomes

$$Ma^2 \rho_* \left(\frac{\partial u_*}{\partial t_*} + u_* \frac{\partial u_*}{\partial r_*} \right) = -\frac{\partial p_*}{\partial r_*}, \tag{6.3}$$

while the conservation equations for mass and energy are

$$\frac{\partial \rho_*}{\partial t_*} + \nabla_* \cdot (\rho_* \boldsymbol{u}_*) = 0, \tag{6.4}$$

$$\frac{\mathrm{d}p_*}{\mathrm{d}t_*} + \gamma p_* \nabla_* \cdot \boldsymbol{u}_* = \gamma D \nabla_* \cdot (K_* \nabla_* T_*). \tag{6.5}$$

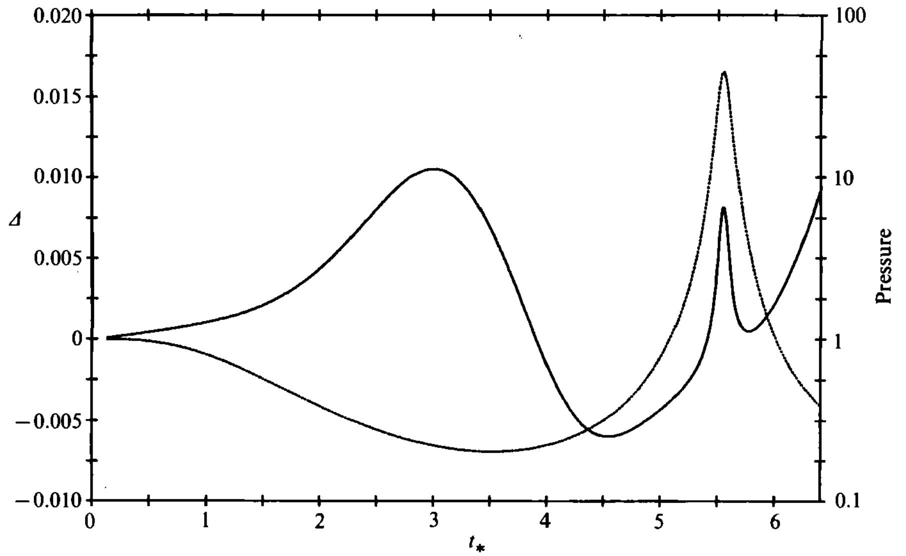

FIGURE 8. The dashed line (right vertical scale) is the internal pressure versus time during the first cycle of the driving pressure predicted by the complete model of §2. the continuous line (left vertical scale) is the pressure difference between the bubble's centre and surface obtained from (6.11). The bubble's equilibrium radius is 100 μm, the frequency $\omega/\omega_0 = 0.8$, and the dimensionless pressure amplitude $\epsilon = 1$. The linear natural frequency is $\omega_0/2\pi = 31.2$ kHz.

If now an expansion in terms of Ma^2 is carried out:

$$p_* = p + Ma^2 p_1 + Ma^4 p_2 + \ldots, \tag{6.6}$$

$$\rho_* = \rho + Ma^2 \rho_1 + \ldots, \tag{6.7}$$

$$u_* = u + Ma^2 u_1 + \ldots, \tag{6.8}$$

we find the momentum equation, to order zero and Ma^2:

$$\frac{\partial p}{\partial r_*} = 0, \tag{6.9}$$

$$-\frac{\partial p_1}{\partial r_*} = \rho\left(\frac{\partial u}{\partial t_*} + u\frac{\partial u}{\partial r_*}\right), \tag{6.10}$$

respectively. Equation (6.9) justifies the assumption made in §2 for Ma^2 small. With this, the energy and continuity equations reduce to the form used in §2 to lowest order in Ma^2.

This formulation can be used to calculate the correction p_1 to the uniform-pressure approximation. Upon integration of the first-order correction (6.10) to the momentum equation from $r_* = 0$ to $r_* = R(t_*)$ we obtain the following expression for the pressure difference between the centre and the wall of the bubble normalized by the lowest-order approximation p obtained from (2.13),

$$\Delta(t_*) \equiv \frac{p_1(R_*(t_*), t_*) - p_1(0, t_*)}{p(t_*)} = -\frac{1}{p(t_*)}\int_0^{R_*(t_*)} \rho\left(\frac{\partial u}{\partial t_*} + u\frac{\partial u}{\partial r_*}\right) \mathrm{d}r_*. \tag{6.11}$$

The continuous line in figure 8 shows a graph of this quantity versus time for a typical case of a bubble with a radius of 100 μm ($\omega_0/2\pi = 31.2$ kHz) driven at a frequency $\omega/\omega_0 = 0.8$ by a sound field with a dimensionless pressure amplitude $\epsilon = 1$. The dashed line (right vertical scale) is a graph of $p(t_*)$, while the continuous line (left vertical scale) shows Δ in the course of the first oscillation. It is seen that this ratio is always very small even though the internal pressure reaches quite high levels. A contributing factor is that, during the phase of more violent motion in which the error tends to increase, the bubble is being compressed so that the temperature rises and with it the speed of sound. The instantaneous value of the Mach number is thereby reduced and the pressure gradients mitigated.

The previous considerations furnish only a partial estimate of the error incurred in assuming a spatially uniform internal pressure. Indeed, from a knowledge of (6.11), no information is obtained on the *absolute level* of the correction p_1, for which the complete first-order problem in Ma^2 must be solved. This consists of the first-order equations for momentum, (6.10), mass, and energy:

$$\frac{\partial \rho_1}{\partial t_*}+\nabla_*\cdot(\rho \boldsymbol{u}_1+\rho_1 \boldsymbol{u}) = 0, \tag{6.12}$$

$$\frac{\partial p_1}{\partial t_*}+u\frac{\partial p_1}{\partial r}+\gamma(p_1\nabla_*\cdot\boldsymbol{u}+p\nabla_*\cdot\boldsymbol{u}_1) = \gamma D\nabla_*\cdot(K_1\nabla_* T+K\nabla_* T_1). \tag{6.13}$$

To these, the equation of state

$$p_1 = \rho T_1+\rho_1 T \tag{6.14}$$

must be added. A solution in the general case can only be obtained numerically. Here we shall consider the linearized problem, which can be handled analytically, and from which some useful indications can be obtained. The equations to be solve are, upon dropping the asterisks for simplicity,

$$\frac{\partial \rho_1}{\partial t}+\nabla\cdot\boldsymbol{u}_1 = 0, \tag{6.15}$$

$$\frac{\partial p_1}{\partial r} = -\frac{\partial u}{\partial t}, \tag{6.16}$$

$$\frac{\partial p_1}{\partial t}+\gamma\nabla\cdot\boldsymbol{u}_1 = \gamma D\nabla^2 T_1, \tag{6.17}$$

$$p_1 = \rho_1+T_1. \tag{6.18}$$

These equations are obtained from (6.10) and (6.12)–(6.14) by neglecting all terms that contain products of first-order quantities in the radial displacement of the bubble interface. In the linear approximation, and for steady oscillations with a unit dimensionless frequency, the non-dimensional form of the velocity (2.4) is

$$u = D\frac{\partial T}{\partial r}-\frac{\mathrm{i}r}{3\gamma}P, \tag{6.19}$$

where P is the dimensionless pressure disturbance defined in (3.3). Upon substitution into (6.16) and integration we find

$$p_1 = -\frac{r^2}{6\gamma}P-\mathrm{i}D(T-1)+C, \tag{6.20}$$

where C is an integration constant. An equation for this quantity can be obtained in the form of a consistency condition by requiring that the first-order correction u_1 to the radial velocity vanish at the bubble boundary. Upon integrating (6.15) and imposing this condition we find

$$\mathrm{i}\int_0^1 r^2 p_1\,\mathrm{d}r = \gamma D\left.\frac{\partial T_1}{\partial r}\right|_1. \tag{6.21}$$

The integral can be calculated by use of (6.20) and the linearized equation (3.1) satisfied by the zero-order temperature field to find

$$C = 3\mathrm{i}DX+\left(\frac{1}{10\gamma}+\mathrm{i}D\right)P-3\mathrm{i}\gamma D\left.\frac{\partial T_1}{\partial r}\right|_1. \tag{6.22}$$

It is now necessary to calculate T_1 to obtain from this equation the final expression for C. To this end we eliminate $\nabla\cdot\boldsymbol{u}_1$ between (6.15) and (6.17) to find

$$\mathrm{i}T_1 = \frac{\gamma-1}{\gamma}\mathrm{i}p_1+D\nabla^2 T_1, \tag{6.23}$$

which is formally the same as (3.1) satisfied by the corresponding lower-order quantities. We substitute (6.20) and solve the equation subject to $T_1 = 0$ at the bubble wall. Upon substitution into (6.22), the following expression for C is found

$$C = \frac{P\left(\mathrm{i}D+\dfrac{1}{10\gamma}+\dfrac{\gamma-1}{2\gamma}\displaystyle\int_0^1 r^2 S(r)\,\mathrm{d}r\right)+3\mathrm{i}DX+3\mathrm{i}(\gamma-1)D\displaystyle\int_0^1 (T-1)\,S(r)\,\mathrm{d}r}{1+3(\gamma-1)\displaystyle\int_0^1 S(r)\,\mathrm{d}r}, \tag{6.24}$$

where $S(r) = r\sinh(r(\mathrm{i}/D)^{\frac{1}{2}})/\sinh(\mathrm{i}/D)^{\frac{1}{2}}$. Since the interest here is to compare the magnitude of p_1 with that of the first-order pressure disturbance P, it is useful to express the dimensionless radius perturbation X in terms of P by use of (3.19). Upon carrying out the integrals, we then have the final result

$$C = \frac{N_C(\mathrm{i}/D)}{D_C(\mathrm{i}/D)}P, \tag{6.25}$$

where

$$N_C(z) = \frac{\gamma}{10}-\frac{\gamma-1}{2\gamma}\left[\frac{5}{z}-6\frac{(3-\gamma)}{z^2}-\frac{3(\gamma-1)}{z\sinh^2 z^{\frac{1}{2}}}+\left(1+3\frac{5-\gamma}{z}\right)\frac{\coth z^{\frac{1}{2}}}{z^{\frac{1}{2}}}\right]$$
$$+\frac{\gamma-1}{2\gamma z^{\frac{1}{2}}}\left[1+3\frac{5-\gamma}{z}\right]\coth z^{\frac{1}{2}},$$

$$D_C(z) = 1+3\frac{\gamma-1}{z}(z^{\frac{1}{2}}\coth z^{\frac{1}{2}}-1).$$

From (6.20), the value of p_1 at the bubble's centre is then

$$\frac{p_1(r=0)}{P} = \frac{\gamma-1}{\gamma}\frac{D}{\mathrm{i}}\left(1-\frac{(\mathrm{i}/D)^{\frac{1}{2}}}{\sinh(\mathrm{i}/D)^{\frac{1}{2}}}\right)+\frac{C}{P}, \tag{6.26}$$

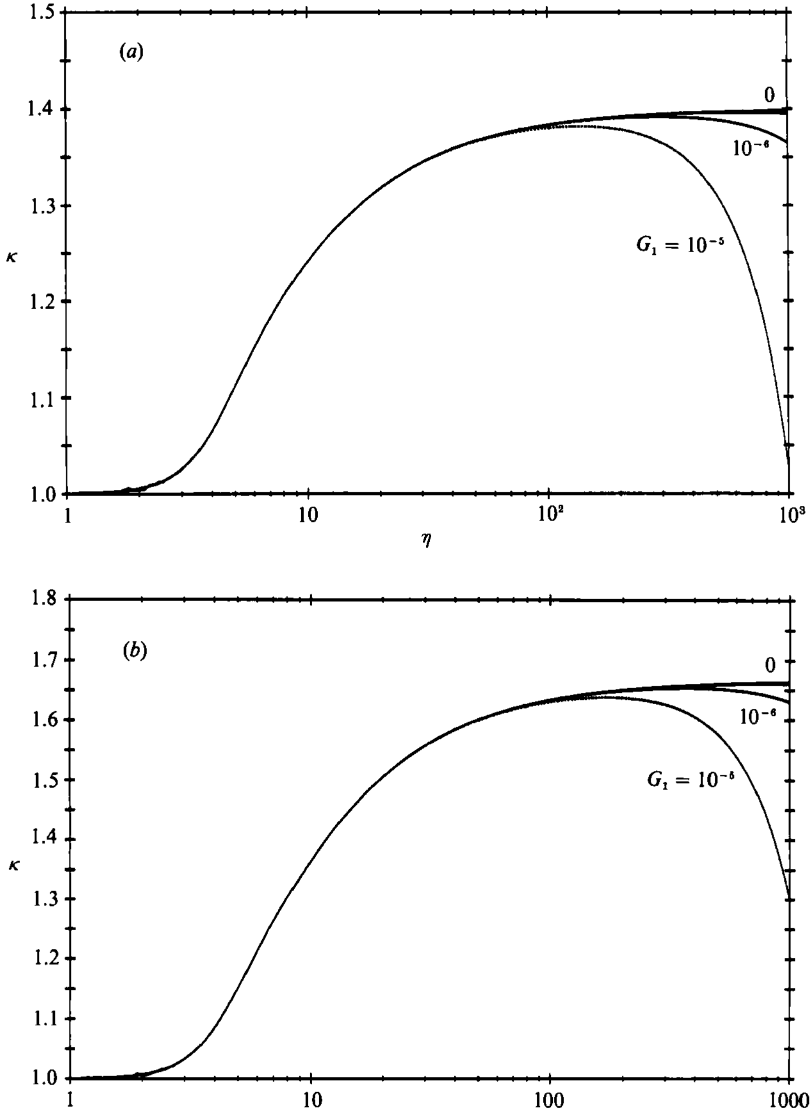

FIGURE 9. The effective polytropic index κ for linear oscillations of figure 1 ($G_1 = 0$) is compared with the results of Prosperetti (1977) obtained without the uniform-pressure approximation for (a) monatomic ($\gamma = \frac{5}{3}$) and (b) diatomic ($\gamma = \frac{7}{5}$) gases. The values of G_1 defined by (6.30) are 10^{-5}, 10^{-6}, 10^{-7}, and 0.

which, for small D, is

$$\frac{p_1(r=0)}{P} \approx \frac{\gamma}{10}, \tag{6.27}$$

while, for large D,

$$\frac{p_1(r=0)}{P} \approx \frac{1}{5}\left(1-\frac{1}{2\gamma^2}\right). \tag{6.28}$$

The two asymptotic limits are numerically quite close, of order 10^{-1} for both $\gamma = \frac{7}{5}$ and $\gamma = \frac{5}{3}$, and the complete function (6.26) varies little between them. These small-amplitude results suggest that the error in the value of the pressure at the bubble's centre introduced by the assumption of uniform pressure is small and does not significantly depend upon the bubble radius or sound frequency. Furthermore, again in the linearized approximation and for sinusoidal oscillations, an analytic expression for the quantity Δ defined in (6.11) is readily obtained,

$$\frac{p_1(R)-p_1(0)}{P} = -\frac{1}{P}\int_0^R \rho\frac{\partial u}{\partial t}\mathrm{d}r = \mathrm{i}\frac{\gamma-1}{\gamma}D\left(1-\frac{(\mathrm{i}/D)^{\frac{1}{2}}}{\sinh(\mathrm{i}/D)^{\frac{1}{2}}}\right)-\frac{1}{6\gamma}, \tag{6.29}$$

a function that goes from $-\frac{1}{6}/\gamma$ at small D to $-\frac{1}{6}$ at large D, with a minimum in between. Hence, the error in the total pressure difference across the bubble is also small.

In this linear case a more direct assessment of the error introduced by the approximation of uniform internal pressure is possible by comparing the expressions of κ and $\mathrm{Im}\tilde{F}$ given in §3 with earlier results (Prosperetti 1977) in which this approximation was avoided and the momentum equation in the gas solved exactly. Those results were presented in terms of two parameters G_1 and G_2 which, in the notation adopted in this paper, are

$$G_1 = \frac{\chi\omega\rho_0}{p_0}, \quad G_2 = \frac{\eta^2}{2\gamma}. \tag{6.30}$$

The parameter G_1 can readily be shown to be of the order of the ratio of the molecular mean free path to the wavelength in the gas, and is therefore very small. The value $G_1 = 0$ corresponds to a spatially uniform pressure. We compare in figures 9(*a*) and 9(*b*) the approximate results of §3 for the polytropic index ($G_1 = 0$) with the more complete ones of Prosperetti (1977) for $G_1 = 10^{-7}$, 10^{-6}, and 10^{-5} for $\gamma = \frac{7}{5}$ and $\frac{5}{3}$. Similar information for $\mathrm{Im}\tilde{F}$ is not presented because the results are indistinguishable from those shown in figure 2. It is clear that the differences between the two results are small and confined to very high frequency. For the case of free oscillations, the results of Prosperetti (1977) coincide with those of Chapman & Plesset (1971) who calculated the natural frequency by a normal-mode approach.

Since, as noted above, the quantity Ma has the physical meaning of the gas Mach number for oscillation amplitudes of the order of the equilibrium radius, the previous considerations do not necessarily extend to the case of the catastrophic collapse of cavitation bubbles during which the radius can undergo an order-of-magnitude decrease. The approximation of uniform pressure may be poor in these conditions.

7. Weakly nonlinear waves in a bubbly liquid

As an application of the theory developed in the preceding sections we consider a simple average-equation model of a bubbly liquid in one space dimension. The model consists of a continuity equation for the liquid phase,

$$\frac{\partial}{\partial t}[\rho_{\mathrm{L}}(1-\alpha)]+\frac{\partial}{\partial z}[\rho_{\mathrm{L}}(1-\alpha)v] = 0, \tag{7.1}$$

a conservation equation for the bubble number density n,

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}(nv) = 0, \tag{7.2}$$

and a momentum equation for the mixture,

$$\rho_{\mathrm{L}}(1-\alpha)\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}\right)+\frac{\partial \hat{P}}{\partial z}=0. \tag{7.3}$$

In these equations α is the gas volume fraction,

$$\alpha=\tfrac{4}{3}\pi R^3 n, \tag{7.4}$$

v is the common (average) velocity of liquid and bubbles, and $\hat{P}$ is the average pressure in the mixture. This is essentially the same as the model proposed by van Wijngaarden (1968), and later re-examined by Caflisch *et al.* (1985). Its validity is restricted to the case of small gas volume fractions, for which the assumption of zero relative velocity is accurate as shown by Caflisch *et al.* (1985). In writing the preceding equations, the assumption has been made that all the bubbles have the same undisturbed radius. For its closure, the model requires an equation for the radial motion of the bubbles. For simplicity, we use the Rayleigh–Plesset equation (2.15) in which the time derivatives are turned into convective derivatives with velocity v and the ambient pressure is identified with the average pressure in the mixture, i.e.

$$R\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2=\frac{1}{\rho_{\mathrm{L}}}\left[p-\frac{2\sigma}{R}-4\frac{\mu_{\mathrm{L}}}{R}\frac{\mathrm{d}R}{\mathrm{d}t}-\hat{P}\right]. \tag{7.5}$$

This identification, which is one of the key aspects of the model, was first introduced by Foldy (1945) in the linear theory of multiple scattering.

We consider the linear case first and we use the earlier result (3.4) to express the internal pressure. A straightforward procedure of elimination leads to the equation

$$3\alpha_0(1-\alpha_0)\frac{\partial^2 X}{\partial t^2}-\frac{p_0}{\rho_{\mathrm{L}}}\frac{\partial^2}{\partial z^2}\left(\omega D\int_0^t F_*(\omega D t')X(z,t-t')\,\mathrm{d}t'-wX\right)$$
$$-4\nu_{\mathrm{L}}\frac{\partial^3 X}{\partial t\,\partial z^2}-R_0^2\frac{\partial^4 X}{\partial t^2\,\partial z^2}+\frac{p_0}{\rho_{\mathrm{L}}}\frac{\partial^2 I}{\partial z^2}=0, \tag{7.6}$$

where w has been defined in (2.20) and, as before, $R=R_0(1+X)$. The quantity $F_*(\tau)$ is the inverse of $\tilde{F}$ given in (3.5) and $I(z,t)$ is defined by

$$I=\int_0^t Q_1(z,t')\,\mathrm{d}t',$$

with Q_1 given in (3.6), and accounts for the initial conditions of the bubble motion. The presence of the convolution integral is a striking feature of this equation that has not been noticed before owing to the oversimplified models of the bubble response contained in earlier models. To establish the connection with more familiar forms, we consider the case in which the relation between P and X can be approximated by the steady-state relation (3.20). As shown by (3.12), this form holds in the nearly isothermal case, and it may be a good approximation for nearly monochromatic waves. With this simplification, the preceding equation becomes

$$\frac{\partial^2 X}{\partial t^2}-\frac{p_0}{\rho_{\mathrm{L}}}\frac{\omega_*}{3\alpha_0(1-\alpha_0)}\frac{\partial^2 X}{\partial z^2}-\frac{2R_0^2}{3\alpha_0(1-\alpha_0)}(\beta_{\mathrm{v}}+\beta_{\mathrm{th}})\frac{\partial^3 X}{\partial t\,\partial z^2}-\frac{R_0^2}{\alpha_0(1-\alpha_0)}\frac{\partial^4 X}{\partial t^2\,\partial z^2}=0, \tag{7.7}$$

where
$$\omega_*^2 = 3\kappa - w, \tag{7.8}$$
is the dimensionless linear frequency. The last two terms, arising from damping and radial inertia, have dispersive character. If they are disregarded for a moment, the velocity of linear waves c_m can be read directly from the equation and is

$$c_m^2 = \frac{p_0(3\kappa - w)}{3\alpha_0(1-\alpha_0)\rho_L}. \tag{7.9}$$

For isothermal bubble motion and no surface tension effects it reduces to the well-known result (van Wijngaarden 1968, 1972)

$$c_m^2 \approx \frac{p_0}{\alpha_0(1-\alpha_0)\rho_L}.$$

To treat the weakly nonlinear case it is useful to introduce dimensionless variables, denoted by an asterisk, by writing

$$z = (c_m/\omega) z_*, \quad n = n_0 n_*, \quad \hat{P} = p_\infty \hat{P}_*,$$
$$v = \alpha_0 c_m v_*, \quad \alpha = \alpha_0 \alpha_*,$$

while the other variables are non-dimensionalized as in §2. We shall also assume that the bubbles' internal pressure may be written as

$$\frac{p}{p_0} = \left(\frac{R_0}{R}\right)^{3\kappa} - \mathcal{N}\left(\frac{R}{R_0}\right), \tag{7.10}$$

where $\mathcal{N}$ is an operator. This form is motivated by the two approximate results (4.9) and (5.8) valid in the nearly isothermal and nearly adiabatic limits, respectively. Upon combining (7.1) and (7.2), the previous model may be written, in dimensionless form,

$$3R_*^2 n_* \frac{dR_*}{dt_*} = \frac{\partial v_*}{\partial z_*}, \tag{7.11}$$

$$\frac{\partial n_*}{\partial t_*} + \alpha_0 \frac{\partial (n_* v_*)}{\partial z_*} = 0, \tag{7.12}$$

$$\frac{\omega_*^2}{3(1-w)} \frac{1-\alpha_0\alpha_*}{1-\alpha_0} \frac{dv_*}{dt_*} = -\frac{\partial \hat{P}_*}{\partial z_*}. \tag{7.13}$$

Furthermore,

$$\Omega^2\left[R_* \frac{d^2R_*}{dt_*^2} + \frac{3}{2}\left(\frac{dR_*}{dt_*}\right)^2\right] = p_* - \frac{w}{R_*} - \frac{2M}{R_*}\frac{dR_*}{dt_*} - (1-w)\hat{P}_*, \tag{7.14}$$

where
$$\Omega^2 = \frac{\omega_*^2}{3(1-\alpha_0)\alpha_0} \frac{R_0^2\omega^2}{c_m^2} \tag{7.15}$$

may be considered a measure of the effect of the inertia of the radial motion.

Let now $\epsilon \ll 1$ be a measure of the strength of the wave and write

$$v_* = \epsilon V_*, \quad \hat{P}_* = 1 + \epsilon \hat{P}_*, \quad R_* = 1 + \epsilon X_*, \quad n_* = 1 + \epsilon N_*. \tag{7.16}$$

Since the model is only applicable for small volume fractions, we can also assume $\alpha_0 \leqslant O(\epsilon)$. To study the weakly nonlinear case, we expand everything in ϵ, retaining terms up to order ϵ^2. To write the resulting equations in a more compact form it is useful to define

$$\frac{1}{\epsilon}\mathcal{N}\left(\frac{R}{R_0}\right) = \mathcal{N}_0 X + \epsilon \mathcal{N}_1 X, \tag{7.17}$$

where $\mathbf{N}_0$ and $\mathbf{N}_1$ are linear operators. With the previous steps we finally find, upon dropping asterisks for convenience,

$$3\frac{\partial X}{\partial t} - \frac{\partial V}{\partial z} = -3\epsilon(N+2X)\frac{\partial X}{\partial t}, \tag{7.18}$$

$$\frac{\partial N}{\partial t} = -\alpha_0 \frac{\partial V}{\partial z}, \tag{7.19}$$

$$\frac{\omega_*^2}{3(1-w)}\frac{\partial V}{\partial t} + \frac{\partial \hat{P}}{\partial z} = 0, \tag{7.20}$$

$$\Omega^2 \frac{\partial^2 X}{\partial t^2} + \omega_*^2 X + \mathcal{N}_0 X + 2M\frac{\partial X}{\partial t} + (1-w)\hat{P} = \epsilon\left\{-\Omega^2\left[X\frac{\partial^2 X}{\partial t^2} + \frac{3}{2}\left(\frac{\partial X}{\partial t}\right)^2\right] + 2\omega_*^2 \zeta X^2 - \mathcal{N}_1 X + 2MX\frac{\partial X}{\partial t}\right\}, \tag{7.21}$$

where
$$\zeta = \frac{1}{2\omega_*^2}\left[\tfrac{3}{2}\kappa(3\kappa+1) - w\right].$$

By considering linear waves progressing to the right with unit dimensionless velocity (i.e. a dimensional velocity equal to c_m) in an otherwise undisturbed mixture, it is easy to show from these relations that, to lowest order,

$$N = 0 + O(\epsilon), \quad \hat{P} - \frac{\omega_*^2}{3(1-w)} V = k + O(\epsilon), \tag{7.22}$$

where k is a constant. From the second of these relations it also follows that, for right-going waves (Broer 1964),

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\left(\hat{P} - \frac{\omega_*^2}{3(1-w)} V\right) = 0 + O(\epsilon^2), \tag{7.23}$$

from which an expression for $\partial V/\partial z$ can be obtained. Upon substitution into (7.18) we find, to a consistent order,

$$\frac{\partial X}{\partial t} - \frac{1-w}{\omega_*^2}\left(\frac{\partial \hat{P}}{\partial t} + 2\frac{\partial \hat{P}}{\partial z}\right) = -2\epsilon X\frac{\partial X}{\partial t}. \tag{7.24}$$

The average pressure $\hat{P}$ can now be taken from the radial equation (7.21) and substituted here to find

$$\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial z}\right)\left(\frac{\Omega^2}{\omega_*^2}\frac{\partial^2 X}{\partial t^2} + X + \frac{2M}{\omega_*^2}\frac{\partial X}{\partial t} + \frac{1}{\omega_*^2}\mathcal{N}_0 X\right) + \frac{\partial X}{\partial t}$$
$$= -2\epsilon X\frac{\partial X}{\partial t} + \frac{\epsilon}{\omega_*^2}\left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial z}\right)\left\{-\Omega^2\left[X\frac{\partial^2 X}{\partial t^2} + \frac{3}{2}\left(\frac{\partial X}{\partial t}\right)^2\right] + 2\zeta\omega_*^2 X^2 - \mathcal{N}_1 X - 2MX\frac{\partial X}{\partial t}\right\}. \tag{7.25}$$

If we only consider right-going waves then, for any quantity q, $(\partial/\partial t+\partial/\partial z)q = O(\epsilon)$. Using this remark, the right-hand side of the preceding equation can be simplified by writing $\partial/\partial t+2\partial/\partial z \approx \partial/\partial z$. Furthermore we introduce a reference frame moving to the right with the velocity of linear waves and defined by $x = z-t$ to find the final result

$$\frac{\partial X}{\partial t}+\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\left[\frac{\Omega^2}{2\omega_*^2}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right)X+\frac{M}{\omega_*^2}X\right]+\frac{1}{2\omega_*^2}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\mathcal{N}_0 X$$
$$= \epsilon(1+\zeta)X\frac{\partial X}{\partial x}-\epsilon\frac{\partial}{\partial x}\left\{\frac{\Omega^2}{2\omega_*^2}\left[X\frac{\partial^2 X}{\partial x^2}+\frac{3}{2}\left(\frac{\partial X}{\partial x}\right)^2\right]+\frac{M}{\omega_*^2}X\frac{\partial X}{\partial x}+\frac{1}{2\omega_*^2}\mathcal{N}_1 X\right\}. \quad (7.26)$$

It is readily checked that, in the linearized limit and with $M = 0$, $\mathcal{N}_0 = 0$, $\mathcal{N}_1 = 0$, this equation is strictly hyperbolic so that initial conditions on X, $\partial X/\partial t$, and $\partial^2 X^2/\partial t^2$ are necessary and sufficient to determine a unique solution. The first two are obvious. The third may be obtained from the initial condition on the pressure field $\hat{P}$ through the radial equation (7.21).

The wave equation (7.26) can be studied in different limits. Here we shall only consider the limit resulting in a modified Korteweg–De Vries equation. To this end we assume that both Ω^2 and damping effects are small, of order ϵ or smaller. Since phenomena in bubbly liquids are typically relatively slow owing to the high inertia and high compressibility, the typical frequency of the wave entering the definition (2.22) of M is likely to be small and $M \ll 1$ is not a very severe restriction. From the explicit expressions given below for $\mathcal{N}_0$, it will be seen that thermal damping is also small for nearly adiabatic or nearly isothermal oscillations. From the expression (7.9) for c_m and the definition (7.15) of Ω, it is seen that this quantity is of the order of $\rho_L(\omega R_0)^2/p_0$, and therefore small at frequencies much smaller than the resonance frequency of the bubbles. One is thus led to the conclusion that the Korteweg–De Vries limit is a meaningful one for (7.26). Phenomena for which other limits might possibly be more appropriate could be strong shock waves, for which the timescales for the bubble motion may be short. In such cases, however, the slip between the bubbles and the liquid could also be important, and the averaged-equation model (7.1)–(7.3) might be invalid.

With the further observation that

$$\left(\frac{\partial}{\partial t}\right)_x = \left(\frac{\partial}{\partial t}\right)_z - \frac{\partial}{\partial z} = O(\epsilon),$$

we obtain from (7.26)

$$\frac{\partial X}{\partial t}-\epsilon(\zeta+1)X\frac{\partial X}{\partial x}+\frac{\Omega^2}{2\omega_*^2}\frac{\partial^3 X}{\partial x^3} = \frac{M}{\omega_*^2}\frac{\partial^2 X}{\partial x^2}-\frac{1}{2\omega_*^2}\frac{\partial}{\partial x}\mathcal{N}_0 X. \quad (7.27)$$

The left-hand side of this equation is in the standard KdV form, while the right-hand side accounts for dissipative effects. As for the operator $\mathcal{N}_0$, in the nearly isothermal case, upon comparison of (7.10) and (7.17) with (4.9), we have $\kappa = 1$ and

$$\mathcal{N}_0 X = \frac{\gamma-1}{5\gamma D}\left(\frac{\partial X}{\partial t}\right)_z = \frac{\gamma-1}{5\gamma D}\left[\left(\frac{\partial X}{\partial t}\right)_x - \frac{\partial X}{\partial x}\right] \approx -\frac{\gamma-1}{5\gamma D}\frac{\partial X}{\partial x}. \quad (7.28)$$

Thus, in this limit, thermal damping results in a term with the same mathematical structure as viscous damping and (7.27) becomes an equation of the Korteweg–De

Vries–Burgers type. In the opposite, nearly adiabatic, limit, from (5.8) we have $\kappa = \gamma$ and

$$\mathcal{N}_0 X = -9\gamma(\gamma-1)\left(\frac{D}{\pi}\right)^{\frac{1}{2}} \int_0^t X(x+t, t-s)\frac{\mathrm{d}s}{s^{\frac{1}{2}}}, \tag{7.29}$$

which could have been written down directly from (3.16).

Following a method proposed by Gorschkov, Ostrovsky & Pelinovsky (1974), one can find an approximate solution of (7.27) in the form of an attenuating soliton. To this end, start with the *ansatz*

$$X = -\frac{3A(t)}{\epsilon(\zeta+1)}\operatorname{sech}^2\left[\frac{\omega_*(A(t))^{\frac{1}{2}}}{2\Omega}\left(x - \int_0^t A(t')\,\mathrm{d}t'\right)\right], \tag{7.30}$$

where $A(t)$ is a slowly varying amplitude function. For A constant, this is the well-known solitary-wave solution of the KdV equation, i.e. (7.27) with a vanishing right-hand side (Whitham 1974). Upon substitution of this expression into (7.27) and integration over x between $-\infty$ and ∞, the second and third terms vanish by symmetry and one is left with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} X^2\,\mathrm{d}x = -\frac{M}{\omega_*^2}\int_{-\infty}^{\infty}\left(\frac{\partial X}{\partial x}\right)^2\mathrm{d}x + \frac{1}{\omega_*^2}\int_{-\infty}^{\infty}\frac{\partial X}{\partial x}\mathcal{N}_0 X\,\mathrm{d}x. \tag{7.31}$$

Upon substitution of (7.30) the first and second integrations can be carried out with the result

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{8MA^2}{15\Omega^2} + \frac{(\surd 2)\epsilon^2(\zeta+1)^2}{36\omega_*\Omega A^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{\partial X}{\partial x}\mathcal{N}_0 X\,\mathrm{d}x. \tag{7.32}$$

In the nearly isothermal case, $\mathcal{N}_0$ is given by (7.28). The integral gives a contribution similar to the viscous one and the final result is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{4}{15}\left(2M + \frac{\gamma-1}{5\gamma D}\right)\frac{A^2}{\Omega^2}, \tag{7.33}$$

which can be integrated immediately to find

$$A(t) = \frac{A(0)}{1 + \dfrac{4}{15}\left(2M + \dfrac{\gamma-1}{5\gamma D}\right)\dfrac{A(0)}{\Omega^2}t}. \tag{7.34}$$

The amplitude of the wave is seen to undergo a decrease at a rate that increases with the initial amplitude. In the nearly adiabatic case it does not seem possible to carry our the integration in closed form and the resulting integro-differential equation for A must be solved numerically. Since, however, phenomena in bubbly liquids tend to occur on timescales slower than the natural period of the individual bubbles, in practice the nearly isothermal case should be encountered more frequently than the nearly adiabatic one.

Some numerical results on the propagation of shock waves in a bubbly mixture on the basis of a model close to that of (7.1)–(7.5) used in this section have been presented in Prosperetti & Kim (1988). The differences between the two models are mainly in the inclusion of liquid compressibility and the use of the linearized form of some convective time derivatives and are therefore minor. Those results demonstrated the major effect of the thermal behaviour of the gas on the structure and evolution of the waves.

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